

SQUARENESS FOR THE MONOPOLE-DIMER MODEL

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ABSTRACT. The monopole-dimer model introduced recently is an exactly-solvable signed generalisation of the dimer model. We show that the partition function of the monopole-dimer model on a graph invariant under a fixed-point free involution is a perfect square. We give a combinatorial interpretation of the square-root of the partition function for such graphs in terms of a monopole-dimer model on a new kind of graph with two types of edges which we call a dicot. The partition function of the latter can be written as a determinant, this time of a complex adjacency matrix. This formulation generalises T. T. Wu's assignment of imaginary orientation for the grid graph to planar dicots. As an application, we compute the partition function for a family of non-planar dicots with positive weights.

1. INTRODUCTION

The dimer model (also known as perfect matchings or 1-factors) for planar graphs is a well-studied exactly solvable statistical-mechanical model whose partition function can be evaluated as a Pfaffian [Kas61]. The generalisation to the monomer-dimer model, long-considered an intractable problem, has been recently solved by Giuliani-Jauslin-Lieb [GJL16]. In another direction, the monopole-dimer model was introduced recently [Ayy15] as a generalisation of the double-dimer model. The partition function of the monopole-dimer model was shown to be given by a determinant for planar graphs. Further, it was found that the partition function of the model on some planar graphs such as the cycle graph C_{4n} and the $2m \times 2n$ grid graph could be expressed as a perfect square.

In this work, we give a combinatorial explanation for the squareness phenomenon for the monopole-dimer model in a general setting using a symmetry argument in the spirit of Jockusch [Joc94] and Ciucu [Ciu97] for the dimer model. To that end, we will define a generalisation of

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graphs with two kinds of edges (solid and dashed), which we will call dicots¹ in Section 2. We will define the monopole-dimer model for dicots and show that the partition function of the model can be written as a determinant in Section 2. When the dicot is planar, we will show that there exists a generalisation of the Kasteleyn orientation using which the model has a natural combinatorial weight.

We note that this formulation also generalises Wu's choice of orientation of $\imath = \sqrt{-1}$ for vertical edges [Wu62] for two-dimensional grid graphs. This trick has proved useful in analysing the dimer model on subgraphs of \mathbb{Z}^2 ; see for example [Ken97].

We will consider graphs with a special automorphism in Section 3 and prove that the monopole-dimer partition function is a square.

In Section 4, we will provide exact results for the partition functions for certain families of dicots and use those results to calculate the free energy in each case. In Sections 4.1 and 4.2, we will explain the squareness for cycles and rectangular grids observed in [Ayy15]. We will then consider dicots on the rectangular grid with additional vertical dashed edges in Section 4.3. In Section 4.4, we will consider a family of non-planar dicots for which the monopole-dimer model is manifestly positive.

2. MONOPOLE-DIMER MODEL ON DICOTS

In this section, we will define dicots, the monopole-dimer model on dicots and prove the result about the partition function of the latter. We then consider the special case of planar dicots, define the Kasteleyn orientation for such dicots and prove the analogous result for the monopole-dimer model there.

Remark on notation: We will always use unaccented symbols for objects associated to graphs and calligraphic symbols for those associated to dicots.

Dicots are generalisations of graphs with two sets of edges. We will denote the two sets of edges on V by A and B . Edges belonging to A will be denoted by solid lines, and those belonging to B , by dashed lines. We will also interchangeably use the term solid (resp. dashed) edges to mean A -type (resp. B -type) edges.

Definition 2.1. *We say that $\mathcal{G} = [V, A, B]$ is a **dicot** if \mathcal{G} satisfies the following conditions.*

- (1) $[V, A \cup B]$ is a bipartite graph.

¹short for *dicotyledons*, a plant whose embryo contains two leaves.

- (2) *The subgraphs $[V, A]$ and $[V, B]$ are simple, i.e. have no loops or multiple edges.*

In other words, there can be at most two edges between any two vertices in \mathcal{G} , and if there are two edges, then both must be of different type. An oriented dicot is one where each solid edge is assigned a direction. We will denote vertex weights by $x(v)$ for $v \in V$ and edge weights as $a(v, v') \equiv a(v', v)$ whenever $(v, v') \in A$ and $b(v, v') \equiv b(v', v)$ whenever $(v, v') \in B$. Dicot with no B -type edges are thus bipartite graphs.

Throughout the paper, graphs and dicots will be undirected, both vertex- and edge- weighted, and having a prescribed orientation (that is an assignment of a direction at each edge). Graphs will be simple (i.e. without loops or multiple edges). The weights are positive real numbers. Unweighted versions are taken care of by setting the weights to be 1. Unless specified explicitly, graphs and dicots with n vertices will have vertex set $V = \{1, \dots, n\}$ and the orientation for graphs (resp. dicots) associated to edges (resp. solid edges) will point from smaller to larger label.

Definition 2.2. *The **complete dicot** on $2n$ vertices, denoted \mathcal{D}_n , is the dicot on $2n$ vertices with all possible solid and dashed edges.*

We now generalise the monopole-dimer model for graphs [Ayy15] to dicots. Recall that monopole-dimer configurations for graphs consist of directed loops of even length including doubled edges with the property that every vertex belongs to either zero or two edges.

Definition 2.3. *A **monopole-dimer configuration** of $\mathcal{G} = [V, A, B]$ with orientation \mathcal{O} is a monopole-dimer configuration of the graph $[V, A \cup B]$ with the additional proviso that every loop must contain an even number of dashed edges. Let $\mathcal{L}(\mathcal{G})$ be the set of monopole-dimer configurations of \mathcal{G} .*

The weight of a monopole-dimer configuration C is given by a formula similar to that of the usual monopole-dimer configuration. Let $C \in \mathcal{L}(\mathcal{G})$ be a monopole-dimer configuration on \mathcal{G} . Decompose C into loops (ℓ_1, \dots, ℓ_q) and isolated vertices (w_1, \dots, w_r) .

We first define the weight of a loop ℓ . Let $\ell = (v_1, \dots, v_{2m})$ with edges e_1, \dots, e_{2m} such that $e_j = (v_j, v_{j+1})$ with $v_{2m+1} \equiv v_1$. Then the

weight of the loop is given by
(2.1)

$$w(\ell) = - \prod_{j=1}^{2m} \begin{cases} +a(v_j, v_{j+1}) & \text{if } e_j \text{ is solid and } v_{j_k} \rightarrow v_{j_k+1} \text{ in } \mathcal{O}, \\ -a(v_j, v_{j+1}) & \text{if } e_j \text{ is solid and } v_{j_k+1} \rightarrow v_{j_k} \text{ in } \mathcal{O}, \\ \imath b(v_j, v_{j+1}) & \text{if } e_j \text{ is a dashed edge,} \end{cases}$$

Notice that the sign of the loop depends on the relative orientations of the solid edges and the number of dashed edges. The weight of the monopole-dimer configuration C is then given by

$$(2.2) \quad \text{wt}(C) = \prod_{j=1}^q w(\ell) \prod_{j=1}^r x(w_j).$$

Definition 2.4. *The **monopole-dimer model** on a dicot \mathcal{G} with orientation \mathcal{O} is the collection $\mathcal{L}(\mathcal{G})$ of dicot monopole-dimer configurations on \mathcal{G} with the weight of each configuration given by (2.2).*

The partition function of the monopole-dimer model on \mathcal{G} is

$$(2.3) \quad \mathcal{Z}(\mathcal{G}) = \sum_{C \in \mathcal{L}(\mathcal{G})} \text{wt}(C).$$

We now define an adjacency matrix for a dicot \mathcal{G} .

Definition 2.5. *The **complex adjacency matrix** $\mathcal{K} \equiv \mathcal{K}(\mathcal{G})$ associated to \mathcal{G} is the matrix indexed by the vertices of \mathcal{G} whose entries are given by*

$$(2.4) \quad \mathcal{K}_{v,v'} = \begin{cases} x(v) & v' = v \\ a(v, v') + \imath b(v, v') & \text{if } v \rightarrow v' \text{ in } \mathcal{O}, \\ -a(v, v') + \imath b(v, v') & \text{if } v' \rightarrow v \text{ in } \mathcal{O}. \end{cases}$$

We will treat $a(v, v')$ (resp. $b(v, v')$) as 0 if v and v' are not connected by an solid (resp. dashed) edge.

Theorem 2.6. *The partition function of the monopole-dimer model on \mathcal{G} is given by*

$$(2.5) \quad \mathcal{Z}(\mathcal{G}) = \det \mathcal{K}(\mathcal{G}).$$

See [Ayy13] for a similar result about the determinant of such matrices.

Proof. Note that if we do not have any dashed edges, \mathcal{G} reduces to a bipartite graph and the proof is a special case of the partition function of the monopole-dimer model on graphs [Ayy15, Theorem 2.5]. Our proof here follows the same strategy.

It should be clear from Definition 2.1(1) and the complex adjacency matrix in Definition 2.5 that terms in the Leibniz expansion of the determinant of $\mathcal{K}(\mathcal{G})$ which contribute correspond to permutations with singletons and even cycles. Since off-diagonal non-zero entries in $\mathcal{K}(\mathcal{G})$ consist of two terms, we also expand the determinant in terms of monomials in these variables. We will first show that these terms are equinumerous with configurations in $\mathcal{L}(\mathcal{G})$. We need to show that the signs are weights of these configurations are counted by the determinant.

We will work in the most general setting of the complete dicot \mathcal{D}_n (see Definition 2.2) with generic vertex- and edge-weights. The result for a generic dicot will follow by setting some of these weights to zero. Let $C \in \mathcal{L}(\mathcal{D}_n)$ be given by $C = (\ell_1, \dots, \ell_m; w_1, \dots, w_{2p})$, where every vertex $v \in V$ is either one of the isolated vertices w_j or belongs to exactly one loop ℓ_j . There will be many configurations with the same isolated vertices and the same loop structure which will also contribute because of various choices of solid and dashed edges, as well as the direction of the loops. Since each of the loops ℓ_j will be summed separately, it suffices to look at a single loop $\ell = (v_1, \dots, v_{2m})$ with edge $e_j = (v_j, v_{j+1})$. The entry corresponding to e_j in $\mathcal{K} \equiv \mathcal{K}(\mathcal{D}_n)$ is then $\mathcal{K}_{v_j, v_{j+1}} = z_j$, a complex number. Then the weight of the reversed edge (v_{j+1}, v_j) , is $\mathcal{K}_{v_{j+1}, v_j} = -\bar{z}_j$, it's negative complex conjugate. If $m = 1$, the contribution of c is $z_1(-\bar{z}_1) = -|z_1|^2$, which is real. If $m \geq 2$, we add the contribution of c to its reverse \hat{c} in $\det \mathcal{K}$, we get

$$z_1 \cdots z_{2m} + (-\bar{z}_1) \cdots (-\bar{z}_{2m}) = 2 \operatorname{Re}(z_1 \cdots z_{2m}).$$

In both cases, we get a real number, which means that the only loops which contribute are those with an even number of dashed edges, and they contribute with factor two precisely when the loop is nontrivial ($m \geq 2$). These are precisely the allowed terms in the monopole-dimer configuration; see Definition 2.3.

We will now show that the signs match. For the loop ℓ considered above, let S be the set of solid edges with $\#S = 2p$, and among these, S_- be those with opposite sign and $\#S_- = s$. Then the sign of the loop is given, according to (2.1), $\operatorname{sgn}(\ell) = (-1)^{m-p+1+s}$. The corresponding term in the determinant is the part in $\operatorname{Re}(\mathcal{K}_{v_1, v_2} \cdots \mathcal{K}_{v_{2m-1}, v_{2m}} \mathcal{K}_{v_{2m}, v_1})$ proportional to $a(e)$ for $e \in S$ and $b(e)$ for $e \notin S$. The sign is then given by $i^{2m-2p}(-1)^s = (-1)^{m-p+s}$ times the sign of the corresponding permutation. Finally, it is well-known that the sign of a permutation can be obtained by assigning a negative sign to each nontrivial cycle. Thus, the sign from the determinant matches that from (2.1). \square

We illustrate Theorem 2.6 in the following example.

Example 2.7. Consider the complete dicot \mathcal{D}_2 . The vertices are labelled $(1, 2, 3, 4)$ and the orientation is the standard one; from lower to higher label. Let the weight on vertex j be x_j and the edge weights be as shown in Figure 1.

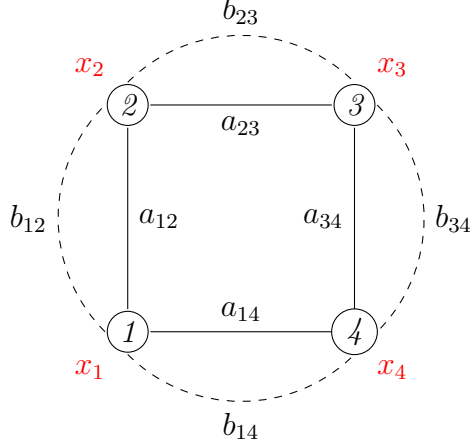


FIGURE 1. The complete dicot \mathcal{D}_2 with generic vertex- and edge-weights.

The complex adjacency matrix in the naturally ordered vertex basis is given by

$$\mathcal{K}(\mathcal{D}_2) = \begin{pmatrix} x_1 & a_{12} + 1b_{12} & 0 & a_{14} + 1b_{14} \\ -a_{12} + 1b_{12} & x_2 & a_{23} + 1b_{23} & 0 \\ 0 & -a_{23} + 1b_{23} & x_3 & a_{34} + 1b_{34} \\ -a_{14} + 1b_{14} & 0 & -a_{34} + 1b_{34} & x_4 \end{pmatrix},$$

and

$$\begin{aligned} \mathcal{Z}(\mathcal{D}_2) &= \det \mathcal{K}(\mathcal{D}_2) \\ &= a_{12}^2 a_{34}^2 + a_{12}^2 b_{34}^2 + a_{34}^2 b_{12}^2 + a_{23}^2 a_{14}^2 + a_{23}^2 b_{14}^2 + b_{23}^2 b_{14}^2 + a_{14}^2 b_{23}^2 + b_{12}^2 b_{34}^2 \\ &\quad + 2a_{12}a_{23}a_{34}a_{14} + 2a_{12}a_{23}b_{34}b_{14} + 2a_{12}a_{34}b_{23}b_{14} + 2a_{23}a_{34}b_{12}b_{14} \\ &\quad - 2a_{12}a_{14}b_{23}b_{34} - 2a_{23}a_{14}b_{12}b_{34} - 2a_{34}a_{14}b_{12}b_{23} - 2b_{12}b_{23}b_{34}b_{14} \\ &\quad + a_{12}^2 x_3 x_4 + a_{23}^2 x_1 x_4 + a_{34}^2 x_1 x_2 + a_{14}^2 x_2 x_3 + b_{12}^2 x_3 x_4 + b_{23}^2 x_1 x_4 \\ &\quad + b_{34}^2 x_1 x_2 + b_{14}^2 x_2 x_3 + x_1 x_2 x_3 x_4. \end{aligned}$$

For the rest of this section, we will consider the special case of a planar dicot.

Definition 2.8. A dicot $\mathcal{G} = [V, A, B]$ is said to be a **planar dicot** if $[V, A \cup B]$ is simple and every face has an even number of dashed edges.

We first define the generalisation of the Kasteleyn orientation of a graph for planar dicots.

Definition 2.9. *Let \mathcal{G} be a planar dicot. Then the **Kasteleyn orientation** \mathcal{O} for \mathcal{G} is one in which the number of clockwise-oriented solid edges plus half the number of dashed edges is odd for every face.*

By considering a spanning tree of the dual graph of planar dicots exactly as for planar graphs, it is easy to see that the following holds.

Proposition 2.10. *Let \mathcal{G} be a planar dicot. Then there exists a Kasteleyn orientation.*

For planar dicots with a Kasteleyn orientation, we give an alternate combinatorial definition of the weight of a loop in the monopole-dimer model.

Corollary 2.11. *Let \mathcal{G} be a planar dicot as above. Then the weight of a loop $\ell = (v_1, \dots, v_{2m})$ in the monopole-dimer model is given by*

$$w(\ell) = (-1)^{\text{number of vertices in } V \text{ enclosed by } \ell} \times \prod_{j=1}^{2m} \begin{cases} a(v_j, v_{j+1}) & \text{if } e_j \text{ is a solid edge,} \\ b(v_j, v_{j+1}) & \text{if } e_j \text{ is a dashed edge.} \end{cases}$$

Proof. We need to show that this definition of the weight is equivalent to that in 2.1. The proof follows *mutatis mutandis* from [Ayy15, Theorem 3.3] by setting o_f to be the number of clockwise-oriented solid edges plus half the number of dashed edges for a face f . \square

Such planar dicots naturally arise when considering subsets of the two-dimensional grid graphs. By setting all vertical edges to be dashed and orienting the solid edges as those in Figure 2, we obtain a Kasteleyn orientation for a large class of planar dicots. Corollary 2.11 extends the observation of Wu [Wu62] about assigning sign i to vertical edges in grid graphs to obtain a Kasteleyn orientation. We generalise this model to grids with both solid and dashed vertical edges in Section 4.3.

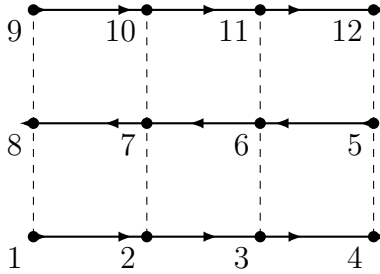


FIGURE 2. A 4×3 “grid dicot” with the Kasteleyn orientation.

A natural question is the following.

Question 2.12. *Classify all dicots \mathcal{G} with arbitrary weights such that $\mathcal{Z}(\mathcal{G})$ is a positive polynomial, i.e. all monopole-dimer configurations contribute with a positive sign.*

We note that planarity is not the correct condition for dicots. For example, no choice of orientation of the solid edges in Example 2.7 will make the partition function $\mathcal{Z}(\mathcal{D}_2)$ a positive polynomial. We will give a nontrivial example of a family of such graphs which are nonplanar in Section 4.4.

3. SQUARENESS FOR THE MONOPOLE-DIMER MODEL

In this section, we will prove that the monopole-dimer model on graphs with a fixed-point-free involution is a perfect square and give a combinatorial interpretation of the square root in terms of a monopole-dimer model on a related dicot.

The monopole-dimer model for graphs G has been defined in [Ayy15]. On bipartite graphs, the monopole-dimer model can be read from Definition 2.4 on a dicot with no dashed, i.e. B -type, edges. In that case, $b(v, v') = 0$ for all $v, v' \in V$. The main result we will need is that the partition function $Z(G)$ for the monopole-dimer model on G can be expressed as the determinant of the (now) real signed adjacency matrix $\mathcal{K}(G)$ given by Definition 2.5.

Recall that an *automorphism* π of a weighted graph $G = [V, E]$ is a bijection from the V to itself which preserves the vertex-edge connectivity as well as vertex- and edge-weights.

Definition 3.1. *Let $G = [V, E]$ be a graph and π be an automorphism of G such that π is a fixed-point free involution. We say that a partition $\mathcal{P} = \{P_1, P_2\}$ of V with $|P_1| = |P_2|$ such that \mathcal{P} is **adapted to π** if the following conditions hold.*

- (1) $v \in P_1$ if and only if $\pi(v) \in P_2$,
- (2) there is no edge between v and $\pi(v)$ for all $v \in P_1$,
- (3) for each edge (v, v') within P_1 , the orientation of the edge (v, v') is the same as that of $(\pi(v), \pi(v'))$, and
- (4) for $v, v' \in P_1$, the edges $(v, \pi(v'))$ and $(v', \pi(v))$ are either both oriented from P_1 to P_2 or both oriented from P_2 to P_1 .

Notice that the last condition in the above definition is consistent with our prescription of orienting edges according to vertex labels since we can choose to label the vertices of P_1 with $\{1, \dots, n\}$ and those of P_2 with $\{n+1, \dots, 2n\}$.

Proposition 3.2. *Let $G = [V, E]$ be a graph and π be an automorphism of G which is a fixed-point free involution. If a partition adapted to π exists, it is unique.*

Proof. Suppose $\mathcal{P} = \{P_1, P_2\}$ and $\mathcal{P}' = \{P'_1, P'_2\}$ are two distinct partitions adapted to π . Since G is connected, there must exist a vertex $v \in P_1 \cap P'_1$ and $w \in P_1 \cap P'_2$ which are connected by an edge. Hence, $\pi(v) \in P_2 \cap P'_2$ and $\pi(w) \in P_2 \cap P'_1$ are also connected by an edge. Without loss of generality, assume that the (v, w) vertex is oriented from v to w . By the adaptedness of \mathcal{P} , the $(\pi(v), \pi(w))$ edge is oriented from $\pi(v)$ to $\pi(w)$, but by the adaptedness of \mathcal{P}' , it is oriented the other way, which is a contradiction. \square

Definition 3.3. *Let $G = [V, E]$ be a graph, π be an automorphism of G which is a fixed-point free involution and \mathcal{P} be a partition adapted to π . Then the **quotient dicot** corresponding to the pair (π, \mathcal{P}) is $\mathcal{D} = G/\pi$ whose solid edges are given by the subgraph of G restricted to P_1 (or equivalently P_2) and whose dashed edges are given by those edges $(u, \pi(v)) \in G$ such that $u \in P_1, v \in P_2$.*

Figure 3 gives an illustration of a graph and its quotient dicot.

Theorem 3.4. *Let G be a bipartite graph satisfying the following conditions with an involution π preserving G and a partition $\mathcal{P} = \{P_1, P_2\}$ of V adapted to π . Consider the quotient dicot $\tilde{\mathcal{G}} = G/\pi$. If $\tilde{\mathcal{G}}$ is also bipartite, then the partition function of the monopole-dimer model on the graph G , $Z(G)$, is given by*

$$Z(G) = Z(\tilde{\mathcal{G}})^2.$$

Proof. We first consider the signed adjacency matrix $K(G)$ with the vertex order $(v_1, \dots, v_n, \pi(v_1), \dots, \pi(v_n))$, where $v_j \in P_1$ and the order of the vertices within P_1 is arbitrary. It is natural to write $K(G)$ in 2×2 block form. Since \mathcal{P} is adapted to π , the $(1, 1)$ and $(2, 2)$ blocks are identical. Moreover, by the fourth condition in Definition 3.1, the $(1, 2)$ block is a symmetric matrix. Thus the signed adjacency matrix can be written as

$$K(G) = \left(\begin{array}{c|c} M & B \\ \hline -B & M \end{array} \right).$$

We now use the identity

$$\begin{pmatrix} \imath/2 \cdot \mathbb{1} & | & -\imath/2 \cdot \mathbb{1} \\ \hline 1/2 \cdot \mathbb{1} & | & 1/2 \cdot \mathbb{1} \end{pmatrix} \cdot \begin{pmatrix} M - \imath B & | & 0 \\ \hline 0 & | & M + \imath B \end{pmatrix} \cdot \begin{pmatrix} -\imath \mathbb{1} & | & \mathbb{1} \\ \hline \imath \mathbb{1} & | & \mathbb{1} \end{pmatrix} \\ = \begin{pmatrix} M & | & B \\ \hline -B & | & M \end{pmatrix},$$

and take the determinant. The determinant of the first and last matrices on the right are easily calculated since the block matrices commute, and we obtain $(i/2)^n$ and $(-2i)^n$ respectively. Thus,

$$(3.1) \quad \det K(G) = \det(M + \imath B) \det(M - \imath B).$$

Let $\mathcal{K}(\tilde{\mathcal{G}})$ be the complex adjacency matrix for $\tilde{\mathcal{G}}$ in the same ordered basis (v_1, \dots, v_n) . We claim that $\mathcal{K}(\tilde{\mathcal{G}}) = M + \imath B$. To see this, first note that as the diagonal entries of B are zero by the second condition in Definition 3.1, the monopole weights are as expected. Moreover, the dashed edges arise exactly as given in Definition 3.3. It therefore remains to show that $\det(M + \imath B)$ is real.

Since $\tilde{\mathcal{G}}$ is also bipartite, we further order the vertices of P_1 as $(p_1, \dots, p_k, p_{k+1}, \dots, p_n)$, where there are no edges among the first k vertices and the last $n - k$ vertices. In this ordering, $M + \imath B$ has the block-diagonal structure

$$\mathcal{K}(\tilde{\mathcal{G}}) = M + \imath B = \begin{pmatrix} N_1 & | & C \\ \hline -C^* & | & N_2 \end{pmatrix},$$

where N_1, N_2 are square diagonal real matrices of sizes $k, n - k$ respectively and C^* is the Hermitian conjugate of C . If some vertex-weights are zero, then either $\det N_1 = 0$ or $\det N_2 = 0$. In that case $\det \mathcal{K}(\tilde{\mathcal{G}}) = \det(CC^*)$, which is real. If all vertex-weights are nonzero, we use the standard formula for the block determinant to get

$$\det \mathcal{K}(\tilde{\mathcal{G}}) = \det(N_1) \det(N_2 + C^* N_1^{-1} C).$$

Since $N_2 + C^* N_1^{-1} C$ is Hermitian, $\det \mathcal{K}(\tilde{\mathcal{G}})$ is real. From (3.1),

$$\det K(G) = \det \mathcal{K}(\tilde{\mathcal{G}}) \overline{\det \mathcal{K}(\tilde{\mathcal{G}})} = \det \mathcal{K}(\tilde{\mathcal{G}})^2,$$

which proves the result. \square

As an application, Theorem 3.4 will be used to justify the squareness of the monopole-dimer model for cycles and rectangular grid proved in [Ayy15] in Sections 4.1 and 4.2. However, we first illustrate the theorem by considering an example.

Example 3.5. Let G be the bipartite graph shown in Figure 3 with edge-weights as shown and vertex weights x_j for $1 \leq j \leq 6$ satisfying $x_j = x_{j+3}$ for $1 \leq j \leq 3$. The map $\pi : j \mapsto j + 3 \pmod 6$ on the vertex set of G then extends naturally to an involution on G . The Partition $\mathcal{P} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$ is adapted to π .

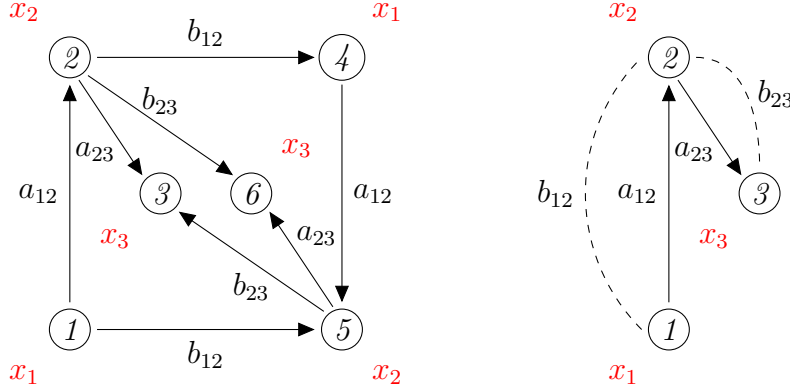


FIGURE 3. The graph G and the dicot quotient \tilde{G} of Example 3.5 with weights and orientations indicated.

The partition function of the monopole-dimer model on G is given by

$$\begin{aligned}
 Z(G) &= \det \begin{pmatrix} x_1 & a_{12} & 0 & 0 & b_{12} & 0 \\ -a_{12} & x_2 & a_{23} & b_{12} & 0 & b_{23} \\ 0 & -a_{23} & x_3 & 0 & b_{23} & 0 \\ 0 & -b_{12} & 0 & x_1 & a_{12} & 0 \\ -b_{12} & 0 & -b_{23} & -a_{12} & x_2 & a_{23} \\ 0 & -b_{23} & 0 & 0 & -a_{23} & x_3 \end{pmatrix} \\
 &= (x_1 x_2 x_3 + a_{12}^2 x_3 + a_{23}^2 x_1 + b_{12}^2 x_3 + b_{23}^2 x_1)^2.
 \end{aligned}$$

The dicot $\tilde{G} = G/\pi$ is shown in Figure 3 with vertex set $\{1, 2, 3\}$.

$$\begin{aligned}
 Z(\tilde{G}) &= \det \begin{pmatrix} x_1 & a_{12} + 1 b_{12} & 0 \\ -a_{12} + 1 b_{12} & x_2 & a_{23} + 1 b_{23} \\ 0 & -a_{23} + 1 b_{23} & x_3 \end{pmatrix} \\
 &= x_1 x_2 x_3 + a_{12}^2 x_3 + a_{23}^2 x_1 + b_{12}^2 x_3 + b_{23}^2 x_1.
 \end{aligned}$$

4. SPECIAL FAMILIES

We consider some families of dicots, calculate the partition function of the monopole-dimer model on them and use the exact results to calculate the free energy.

4.1. Cycle dicots. Theorem 3.4 gives an explanation for the squareness of the monopole-dimer model on the cycle graph C_{4n} with vertex weights x and edge-weights a . It was shown in [Ayy15, Example 3.4] that

$$Z(C_{4n}) = \left(a^{2n} L_{2n}(x/a)\right)^2,$$

where $L_n(x)$ is the Lucas polynomial defined by the recurrence $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$ with initial conditions $L_0(x) = 2, L_1(x) = x$.

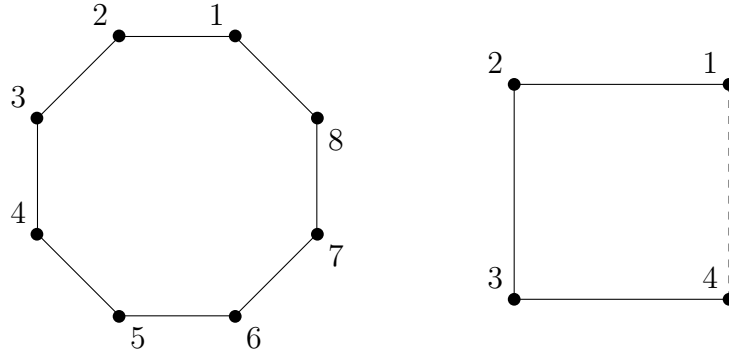


FIGURE 4. The cycle graph C_8 on the left and the cycle dicot C_4 on the right. In both, the orientations are from lower to higher label.

Consider the cycle dicot \mathcal{C}_{2n} on vertices $\{1, \dots, 2n\}$ where the edges $(j, j+1)$ are solid for $1 \leq j \leq 2n-1$ and the sole dashed edge is $(1, 2n)$. See Figure 4 for an illustration for the case $n = 2$. Then the partition function for the monopole-dimer model on this cycle dicot is

$$\mathcal{Z}(\mathcal{C}_{2n}) = \det \begin{pmatrix} x & a & 0 & \cdots & 0 & \imath a \\ -a & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & a \\ \imath a & 0 & \cdots & 0 & -a & x \end{pmatrix}.$$

By writing down a recurrence and using identities relating Lucas and Fibonacci polynomials, one can show that the right hand side is given by $a^{2n} L_{2n}(x/a)$ in agreement with Theorem 3.4.

It is well-known (see for example [HB73]) that the Lucas polynomials can be written as

$$L_n(x) = \prod_{j=0}^{n-1} \left(x - 2i \cos \frac{(2j+1)\pi}{2n} \right).$$

Using this expression, we can calculate the free energy

$$\begin{aligned} F(\mathcal{C}) &:= \lim_{n \rightarrow \infty} \frac{1}{2n} \log a^{2n} L_{2n} \left(\frac{x}{a} \right) \\ &= \log a + \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{j=0}^{n-1} \log \left(\frac{x^2}{a^2} + 4 \cos^2 \frac{(2j+1)\pi}{2n} \right). \end{aligned}$$

Converting the last sum to a Riemann integral, we obtain

$$F(\mathcal{C}) = \log a + \int_0^1 \log \left(\frac{x^2}{a^2} + 4 \cos^2 \frac{\pi t}{2} \right) dt,$$

which can be evaluated exactly, leaving us with the simple expression

$$F(\mathcal{C}) = \log \left(\frac{x + \sqrt{x^2 + 4a^2}}{2} \right).$$

4.2. Rectangular grid graphs. Let $Q_{2m,2n}$ be the $2m \times 2n$ rectangular grid graph with horizontal (resp. vertical) edge-weights a (resp. b) and vertex-weights x . $Q_{2m,2n}$ inherits the fixed-point free involution π which maps vertex (j, k) to $(2m+1-j, 2n+1-k)$ for $1 \leq j \leq 2m, 1 \leq k \leq 2n$.

We consider the orientation \mathcal{O} on $Q_{2m,2n}$ given as follows: $(j, k) \rightarrow (j+1, k)$ if k is even and the other way if k is odd, and $(j, k) \rightarrow (j, k+1)$ if $k < n$ and the other way if $k \geq n$. It is easy to see that \mathcal{O} is a Kasteleyn orientation [Kas61].

For the monopole-dimer model on the $2m \times 2n$ rectangular grid, the partition function had an explicit form.

Theorem 4.1 ([Ayy15, Theorem 4.1]). *The partition function of the monopole-dimer model on the graph $Q_{2m,2n}$ is given by*

$$\prod_{j=1}^m \prod_{k=1}^n \left(x^2 + 4a^2 \cos^2 \left(\frac{j\pi}{2m+1} \right) + 4b^2 \cos^2 \left(\frac{k\pi}{2n+1} \right) \right)^2.$$

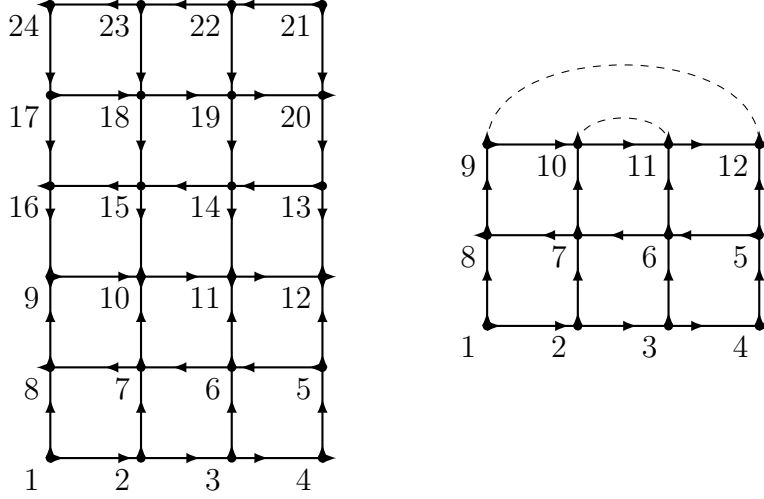


FIGURE 5. Illustration of $Q_{4,6}$ on the left and $\tilde{Q}_{4,3}$ on the right along with orientations.

Then one can check from Definition 3.1 that $\mathcal{P} = \{P_1, P_2\}$ with

$$P_1 = \{(j, k) \mid 1 \leq j \leq 2m, 1 \leq k \leq n\} \quad \text{and} \\ P_2 = \{(j, k) \mid 1 \leq j \leq 2m, n+1 \leq k \leq 2n\},$$

is an adapted partition. Then define the quotient dicot $\tilde{Q}_{2m,n} = Q_{2m,2n}/\pi$ by Definition 3.3. An illustration of the dicot $\tilde{Q}_{4,3}$ is given in Figure 5. We are now in a position to apply Theorem 3.4 and we thus obtain the following corollary.

Corollary 4.2. *The partition function of the monopole-dimer model on the dicot $\tilde{Q}_{2m,n}$ is given by*

$$\prod_{j=1}^m \prod_{k=1}^n \left(x^2 + 4a^2 \cos^2 \left(\frac{j\pi}{2m+1} \right) + 4b^2 \cos^2 \left(\frac{k\pi}{2n+1} \right) \right).$$

The computation of the free energy $F(\tilde{Q})$ has already been performed in [Ayy15, Section 5].

4.3. Rectangular grid dicots. We now consider the monopole-dimer model on the dicot $Q_{m,n}^{(v)}$, which has the same vertex set as and includes all the edges of $Q_{m,n}$ but has in addition a dashed edge for every vertical solid edge. Horizontal edges are assigned weight a , vertical solid edges, b_1 , vertical dashed edges, b_2 , and vertices x . Note that not all configurations have positive weight; see the example in Figure 6. Set $|b| = \sqrt{b_1^2 + b_2^2}$.

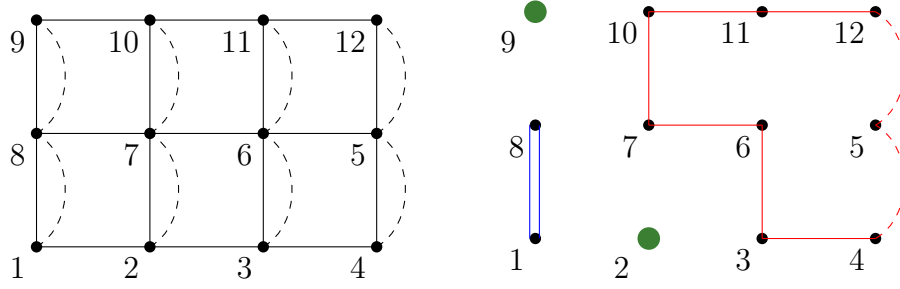


FIGURE 6. Illustration of the dicot $\mathcal{Q}_{4,3}^{(v)}$ on the left and a particular monopole-dimer configuration on the right with weight $-a^4 b_1^4 b_2^2 x^2$.

Define the function

$$Y_m(b, c; x) = \prod_{j=1}^{\lfloor m/2 \rfloor} \left(x^2 + 4(b^2 + c^2) \cos^2 \left(\frac{j\pi}{m+1} \right) \right).$$

Then we have the following result.

Theorem 4.3. *The partition function of the monopole-dimer model on $\mathcal{Q}_{m,n}^{(v)}$ is given by*

$$(4.1) \quad \mathcal{Z}_{m,n}^{(v)} = \prod_{j=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{\lfloor n/2 \rfloor} \left(x^2 + 4a^2 \cos^2 \left(\frac{j\pi}{m+1} \right) + 4|b|^2 \cos^2 \left(\frac{k\pi}{n+1} \right) \right)^2 \\ \times \begin{cases} 1 & \text{if } m \text{ and } n \text{ are even,} \\ Y_m(a, 0; x) & \text{if } m \text{ is even and } n \text{ is odd,} \\ Y_n(b_1, b_2; x) & \text{if } m \text{ is odd and } n \text{ is even,} \\ xY_m(a, 0; x)Y_n(b_1, b_2; x) & \text{if } m \text{ and } n \text{ are odd.} \end{cases}$$

Proof. The strategy of diagonalisation is very similar to that of [Ayy15, Theorem 4.1]. The sole difference is in the matrices involved in the diagonalisation process. Consider the $n \times n$ tridiagonal Toeplitz matrix

given by

$$T_n(c_{-1}, c_0, c_1) = \begin{pmatrix} c_0 & c_1 & 0 & \cdots & 0 & 0 \\ c_{-1} & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & c_1 \\ 0 & 0 & \cdots & 0 & c_{-1} & c_0 \end{pmatrix}.$$

It is well-known (see for example [Tre85]) that the eigenvalues of $T_n(c_{-1}, c_0, c_1)$ are

$$\lambda_q = c_0 + 2\sqrt{c_1 c_{-1}} \cos \frac{q\pi}{n+1}, \quad q = 1, \dots, n,$$

and the matrix of column eigenvectors is

$$u_n(c_{-1}, c_1)_{j,q} = \left(\frac{c_{-1}}{c_1} \right)^{(j-1)/2} \sin \frac{qj\pi}{n+1}, \quad j = 1, \dots, n,$$

respectively. Let J_m be the antidiagonal matrix of ones of size m . The complex adjacency matrix with the standard ordering (as in Figure 6) can be written using tensor notation as

$$\mathcal{K}_{m,n}^{(v)} = \mathbb{1}_n \otimes T_m(-a, x, a) + T_n(-b_1 + \imath b_2, 0, b_1 + \imath b_2) \otimes J_m.$$

Now we perform a similarity transformation using $u_n(-b_1 + \imath b_2, b_1 + \imath b_2) \otimes u_m(-a, a)$ leading to a block diagonal matrix whose k 'th block is the cruciform matrix

$$\begin{pmatrix} x + 2\imath a \cos \frac{\pi}{m+1} & & & (-1)^{m-1} 2\imath^m |b| \cos \frac{\pi k}{n+1} \\ & \ddots & \ddots & \\ & \ddots & \ddots & \\ (-1)^0 2\imath^m |b| \cos \frac{\pi k}{n+1} & & & x + 2\imath a \cos \frac{m\pi}{m+1} \end{pmatrix}.$$

The determinant of the block-diagonal matrix is thus easily computed. The four cases where m and n are of different parities are handled exactly as in the proof of [Ayy15, Theorem 4.1]. \square

When we set $b_1 = 0$, we obtain a planar dicot for which we can define a Kasteleyn orientation using Proposition 2.10. This then matches the orientation defined by Wu [Wu62].

The free energy of the monopole-dimer model on the infinite grid dicot with vertical dashed edges $\lim_{m,n \rightarrow \infty} \mathcal{Q}_{m,n}^{(v)}$ is then given by

$$\begin{aligned} F(\mathcal{Q}^{(v)}) &= \lim_{m,n \rightarrow \infty} \frac{1}{mn} \log \mathcal{Z}_{m,n}^{(v)} \\ &= \frac{2}{\pi^2} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \ln (x^2 + 4a^2 \cos^2 \theta + 4|b|^2 \cos^2 \phi), \end{aligned}$$

and the results are analogous to that of the monopole-dimer model on the grid graph described in [Ayy15, Section 5] with b there replaced by $|b|$.

4.4. Wheel dicots. For n an odd integer, define the *wheel dicot of order n* , denoted \mathcal{W}_n , to be a dicot on the vertex set $V = [2n]$ with solid edges such that the graph $[V, A]$ is the ordinary cycle graph C_{2n} and dashed edges connecting antipodal vertices. It is easy to see that \mathcal{W}_n satisfies the conditions of Definition 2.1.

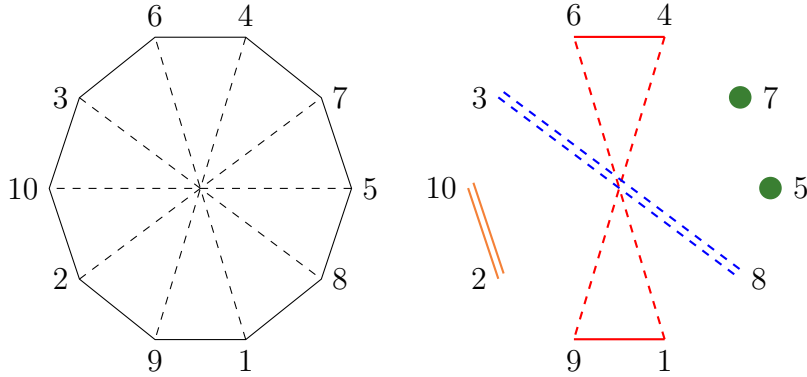


FIGURE 7. The wheel dicot \mathcal{W}_5 on the left and a particular monopole-dimer model configuration on the right with weight $x^2 a^4 b^4$.

We now label the vertices of \mathcal{W}_n as follows starting by labelling an arbitrary vertex 1. Proceeding clockwise, we label every alternate vertex with the next integer until we assign label n . We now label the vertex antipodal to 1 by $n + 1$. Similarly, proceeding clockwise, we label every alternate vertex with the next integer until we assign label $2n$. At this point we have labelled every vertex. See Figure 7 for the dicot \mathcal{W}_5 with its labelling. We then assign the canonical orientation on solid edges. Thus, each vertex is oriented either outwards or inwards on both edges.

Consider the monopole-dimer model on \mathcal{W}_n , where vertex-weights are x , solid edge-weights are a and dashed edge-weights are b . Then we have the following result.

Lemma 4.4. *Every monopole-dimer configuration on \mathcal{W}_n has a positive weight.*

Proof. Since the weights of doubled edges and isolated vertices have positive sign, it suffices to prove that every loop has positive weight. We will divide the proof in two parts. We will first show that every nontrivial loop involving dashed edges is 8-shaped with exactly two dashed edges. See the example monopole-dimer model configuration in Figure 7 for an illustration. We will then show that all loops have positive weight.

The first statement follows from the fact that every loop contains an even number of dashed edges. For example, suppose we have a loop in \mathcal{W}_5 of Figure 7 starting with $6 - 4 - 9 - 2 - \dots$. To eventually get back to 6, we need an even number of dashed edges to reach the same side of the $4 - 9$ dashed edge as 6. But this would imply that we have an odd number of total dashed edges. Thus, a loop starting $6 - 4 - 9$ has to extend as $6 - 4 - 9 - 1 - \dots$. It is now easy to see that there has to be exactly one more dashed edge in the loop.

Recall the formula for the weight of a loop, (2.1). First consider (trivial) loops of length 2 with $m = 1$. Such loops have positive weight in both solid and dashed cases. The only nontrivial loops without dashed edges are the circumferences. In this case, $m = p = n$, and there are n negative orientations, leading to a positive weight.

Lastly, for 8-shaped loops of length $2m$, $m = p - 1$ and hence the prefactor is $+1$. We thus have to show that the sign coming purely from orientations of solid edges is $+1$. First, note that will have $m - 1$ solid edges on either side. If m is odd, then there are $(m - 1)/2$ edges of opposite orientation on either side leading to a total sign of $(-1)^{m-1} = +1$. If m is even, we have either $(m - 2)/2$ or $m/2$ edges with opposite orientation on one side. But because dashed edges connect vertices on different (bipartite) parts, we will have the same number of edges with opposite orientation on the other side too, leading to a total sign of either $(-1)^{m-2}$ or $(-1)^m$, which is again $+1$. \square

Theorem 4.5. *The partition of the monopole-dimer model on the wheel dicot \mathcal{W}_n for odd n is given by*

$$\begin{aligned}\mathcal{Z}(\mathcal{W}_n) &= \prod_{j=0}^{n-1} \left(x^2 + b^2 + 4a^2 \cos^2 \frac{\pi j}{n} \right) \\ &= (x^2 + b^2 + 4a^2) \prod_{j=1}^{(n-1)/2} \left(x^2 + b^2 + 4a^2 \cos^2 \frac{\pi j}{n} \right)^2.\end{aligned}$$

Proof. We will use Theorem 2.6 to calculate $\mathcal{Z}(\mathcal{W}_n)$. The complex adjacency matrix $\mathcal{K}(\mathcal{W}_n)$ using the natural order of the vertices is given by

$$\mathcal{K}(\mathcal{W}_n) = \left(\begin{array}{c|c} x\mathbb{1} & aA + \imath b\mathbb{1} \\ \hline -aA + \imath b\mathbb{1} & x\mathbb{1} \end{array} \right),$$

where A is an $n \times n$ symmetric $(0, 1)$ -band matrix consisting of two consecutive bands above and below the diagonal equidistant from both ends. Since both the matrices in the top-blocks commute, the determinant of $\mathcal{K}(\mathcal{W}_n)$ can be written in a way analogous to a 2×2 matrix as

$$(4.2) \quad \det \mathcal{K}(\mathcal{W}_n) = \det(x\mathbb{1} \cdot x\mathbb{1} - (aA + \imath b\mathbb{1}) \cdot (-aA + \imath b\mathbb{1}))$$

$$(4.3) \quad = \det((x^2 + b^2)\mathbb{1} + a^2 A^2).$$

It now suffices to calculate the eigenvalues of A^2 . But this is easily done since A^2 is of the form

$$A^2 = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix},$$

and hence is a circulant matrix. Thus, the eigenvalues of A^2 are given by

$$\lambda_j = 2 + \omega_j + \omega_j^{n-1}, \quad \text{for } j = 0, 1, \dots, n-1,$$

where $\omega_j = \exp(2\pi \imath j/n)$ is the n 'th root of unity. It is easy to see that

$$\lambda_j = 4 \cos^2 \frac{\pi j}{n}.$$

Since the determinant of $\mathcal{K}(\mathcal{W}_n)$ is the product of its eigenvalues, we use this expression in (4.2) to complete the proof. \square

From Theorem 4.5, one can also calculate the free energy of the monopole-dimer model for wheel dicots. Since the relevant parameter is the ratio of $4a^2$ and $x^2 + b^2$, we set the former to α and the latter to 1. The free energy is

$$\begin{aligned} F(\mathcal{W}) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{Z}(\mathcal{W}_n), \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left(1 + \alpha \cos^2 \frac{\pi j}{n} \right). \end{aligned}$$

We replace the right hand side by the Riemann integral to obtain

$$F(\mathcal{W}) = \int_0^1 \log (1 + \alpha \cos^2(\pi t)) dt,$$

which is essentially the same integral that we saw for the free energy of cycle dicots in Section 4.1. Performing the integral, we obtain

$$F(\mathcal{W}) = 2 \log \left(\frac{1 + \sqrt{1 + \alpha}}{2} \right).$$

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